Tutorial 11

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1 Question 1: §9.1 Q2

Show that if a series is conditionally convergent, then the series obtained from its positive terms is divergent, and the series obtained from its negative terms is divergent.

Proof. Recall the definition of conditionally convergence: A series is said to be conditionally convergent if it is convergent, but it is not absolutely convergent.

Let $\{a_n\}$ be a conditionally convergent series and $a_n = p_n - q_n$. $p_n = a_n$ if $a_n > 0$ or 0 if $a_n \le 0$. $q_n = -a_n$ if $a_n < 0$ or 0 if $a_n \ge 0$.

Then the convergence of positive terms is same as the convergence of $\{p_n\}$. The convergence of negative terms is same as the convergence of $\{q_n\}$.

Without loss any generality, let $\{p_n\}$ be convergent.

Claim $\{p_n - a_n\}$ is convergent. Since $\{p_n\}$ and $\{a_n\}$ are convergent, for any $\varepsilon > 0$, there exists N_1 and N_2, such that

$$|x_m + x_{m+1} + \cdots + x_n| \le \varepsilon/2$$
$$|p_m + p_{m+1} + \cdots + p_n| \le \varepsilon/2$$

for any $n > m > \max\{N_1, N_2\}$.

Hence,

$$\begin{aligned}&|p_m + p_{m+1} + \cdots + p_n - (x_m + x_{m+1} + \cdots + x_n)|\\ \le&|x_m + x_{m+1} + \cdots + x_n| + |p_m + p_{m+1} + \cdots + p_n| \le \varepsilon\end{aligned}$$

for any $n > m > \max\{N_1, N_2\}$. Thus, $\{p_n - a_n\}$ is convergent, which means $\{q_n\}$ is convergent. Hence, $\{p_n + q_n\}$ is convergent, which is absolutely convergent, which means $\{|a_n|\}$ is absolutely convergent, a contradiction. □

2 Question 2: §9.2 Q3(e)

Discuss the convergence or the divergence of the series with n-th term given by $(n \ln n)^{-1}$.

Proof. By Integral Test 9.2.6, let $f(x) = \frac{1}{x \ln x}$, decreasing function on $\{t \ge 2\}$. Then $\sum_2^\infty f(n)$ converges if and only if $\int_2^\infty f(x)dx$ exists.

Since $\int_2^\infty f(x)dx = \ln(\ln x)|_2^\infty = \infty$, hence $\sum_2^\infty f(n)$ is divergent. □

3 Question 3: §9.2 Q4(f)

Discuss the convergence or the divergence of the series with n-th term given by $\frac{n!}{e^{-n^2}}$.

Proof. By Stirling's approximation, $n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$, hence the original series approximates $\sqrt{2\pi n}\frac{n^n}{e^{n+n^2}}$.

Then

$$\lim|\sqrt{2\pi n}\frac{n^n}{e^{n+n^2}}|^{1/n} = \lim(\sqrt{2\pi n})^{1/n}\frac{n}{e^{n+1}} = 0$$

Hence, it is convergent. □

4 Question 4: §9.2 Q7(b)

Discuss the convergence or the divergence of the series with n-th term given by $\frac{n!}{(2n!)}$.

Proof. By Stirling's approximation, $n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$, hence the original series approximates

$$\frac{\sqrt{2\pi n}}{\sqrt{4\pi n}}\frac{n^n}{(2n)^{2n}}\frac{e^{2n}}{e^n} = \frac{1}{\sqrt{2}}\frac{e^n}{(4n)^n}.$$

What's more,

$$\lim(x_n)^{1/n} = \lim\frac{1}{2^{1/2n}}\frac{e}{4n} = 0$$

Hence, it is convergent. □

5 Question 5: §9.3 Q8(b)(c)

Discuss the series whose n-th term is:

(b) $\frac{n^n}{(n+1)^{n+1}}$

(c) $(-1)^n\frac{(n+1)^n}{(n^n)}$.

Proof. (b) It is clear to see $\frac{n^n}{(n+1)^{n+1}} = \frac{1}{n+1}(\frac{n}{n+1})^n = \frac{1}{n+1}\frac{1}{(1+\frac{1}{n})^n}$. From Question 2, there exists $N > 0$ such that

$$ln(n+1) > (1+\frac{1}{n})^n$$

for $n > N$, since $\ln n+1$ and $(1+\frac{1}{n})^n$ are monotone increasing sequences and $\lim\ln n = \infty$ and $\lim(1+\frac{1}{n})^n = e$.

Hence $\frac{1}{n\ln n} \leq \frac{1}{n+1}\frac{1}{(1+\frac{1}{n})^n}$, for $n > N$. Hence it is divergent due to the divergence of $\sum \frac{1}{n\ln}$.

(c) By 3.7.3, The n-th Term test shows that if a series $\sum x_n$ converges, then $\lim x_n = 0$. Since $\lim \frac{(n+1)^n}{n^n} = e$, thus it is divergent. □